

Monogamy of α th Power Entanglement Measurement in Qubit Systems

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In this paper, we study the α th power monogamy properties related to the entanglement measure in bipartite states. The monogamy relations related to the α th power of negativity and the Convex-Roof Extended Negativity are obtained for N -qubit states. We also give a tighter bound of hierarchical monogamy inequality for the entanglement of formation. We find that the GHZ state and W state can be used to distinguish the α th power the concurrence for $0 < \alpha < 2$. Furthermore, we compare concurrence with negativity in terms of monogamy property and investigate the difference between them.

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I. INTRODUCTION

Multipartite entanglement is an important physical resource in quantum mechanics, which can be used in quantum computation, quantum communication and quantum cryptography. One of the most surprising phenomenon for multipartite entanglement is the monogamy property, which may be as fundamental as the no-cloning theorem [1–4]. The monogamy property can be interpreted as the amount of entanglement between A and B , plus the amount of entanglement between A and C , cannot be greater than the amount of entanglement between A and the pair BC . Monogamy property have been considered in many areas of physics: one can estimate the quantity of information captured by an eavesdropper about the secret key to be extracted in quantum cryptography [3, 5], the frustration effects observed in condensed matter physics [6], even in black-hole physics [7, 8].

Historically, monogamy property of various entanglement measure have been discovered. Coffman *et al* first considered three qubits A, B and C which may be entangled with each other [2], who showed that the squared concurrence C^2 follows this monogamy inequality. Osborne *et al* proved the squared concurrence follows a general monogamy inequality for N -qubit system [3]. Analogous to the Coffman-Kundu-Wootters (CKW) inequality, Ou *et al* proposed the monogamy inequality holds in terms of squared negativity \mathcal{N}^2 [10]. Kim *et al* showed that the squared convex-roof extended negativity $\tilde{\mathcal{N}}^2$ follows monogamy inequality [11]. Oliveira *et al* and Bai *et al* investigated entanglement of formation (EoF) and showed that the squared EoF E^2 follows the monogamy inequality [12, 13]. A natural question is why those monogamy property above are squared entanglement measure? In fact, Zhu *et al* showed that the α th power of concurrence C^α ($\alpha \geq 2$) and the α th power of entanglement of formation E^α ($\alpha \geq \sqrt{2}$) follow the

general monogamy inequalities [14].

In this paper, we study the monogamy relations related to α th power of some entanglement measures. We show that the α th power of negativity \mathcal{N}^α and the α th power of convex-roof extended negativity (CREN) $\tilde{\mathcal{N}}^\alpha$ follows the hierarchical monogamy inequality for $\alpha \geq 2$ [15]. From the hierarchical monogamy inequality, the general monogamy inequalities related to \mathcal{N}^α and $\tilde{\mathcal{N}}^\alpha$ are obtained for N -qubit states. We find that the GHZ state and W state can be used to distinguish the \mathcal{C}^α for $0 < \alpha < 2$, which situation was not clear in Zhu *et al*'s paper [14]. The hierarchical monogamy inequality for E^α is also discussed, which improved Bai *et al*'s result [13, 15].

This paper is organized as follows. In Sec. II, we study the monogamy property of α th power of negativity. In Sec. III, we discuss the monogamy property of α th power of CREN. In Sec. IV, we study the monogamy property of α th power of EoF. In Sec. V, we compare the monogamy property of concurrence with negativity. We summarize our results in Sec. VI.

II. MONOGAMY OF α TH POWER OF NEGATIVITY

Given a bipartite state ρ_{AB} in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Negativity is defined as [9]:

$$\mathcal{N}(\rho_{AB}) = \frac{\|\rho_{AB}^{T_A}\| - 1}{2}, \quad (1)$$

where $\rho_{AB}^{T_A}$ is the partial transpose with respect to the subsystem A , $\|X\|$ denotes the trace norm of X , i.e. $\|X\| \equiv \text{Tr} \sqrt{XX^\dagger}$. Negativity is a *computable* measure of entanglement, and which is a convex function of ρ_{AB} . $\mathcal{N}(\rho_{AB}) = 0$ if and only if ρ_{AB} is separable for the $2 \otimes 2$ and $2 \otimes 3$ systems [21]. For the purposes of discussion, we use following definition of negativity:

$$\mathcal{N}(\rho_{AB}) = \|\rho_{AB}^{T_A}\| - 1. \quad (2)$$

For any maximally entangled state in two-qubit system, this definition of negativity is equal to 1.

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For a bipartite pure state $|\psi_{AB}\rangle$, the concurrence is defined as:

$$\mathcal{C}(|\psi_{AB}\rangle) = \sqrt{2[1 - \text{Tr}(\rho_A^2)]} = 2\sqrt{\det \rho_A}, \quad (3)$$

where ρ_A is the reduced density matrix of subsystem A. For a mixed state ρ_{AB} , the concurrence can be defined as:

$$\mathcal{C}(\rho_{AB}) = \min \sum_i p_i \mathcal{C}(|\psi_{AB}^i\rangle), \quad (4)$$

where the minimum is taken over all possible pure state decompositions $\{p_i, |\psi_{AB}^i\rangle\}$ of ρ_{AB} .

The next *lemma* builds a relationship between negativity and concurrence in a $2 \otimes m \otimes n$ system ($m \geq 2, n \geq 2$).

Lemma 1 . For a pure state $|\psi\rangle_{ABC}$ in a $2 \otimes m \otimes n$ system ($m \geq 2, n \geq 2$), the negativity of bipartition $A|BC$ is equal to its concurrence: $\mathcal{N}_{A|BC} = \mathcal{C}_{A|BC}$, where $\mathcal{N}_{A|BC} = \mathcal{N}(|\psi_{ABC}\rangle)$ and $\mathcal{C}_{A|BC} = \mathcal{C}(|\psi_{ABC}\rangle)$.

Proof: Based on the Schmidt decomposition, we can write the bipartition $A|BC$ as: $|\psi_{A|BC}\rangle = \sum_i \sqrt{\lambda_i} |\phi_A^i\rangle \otimes |\phi_{BC}^i\rangle$, where λ_i are Schmidt coefficients and $\sum_i \lambda_i = 1$. $\{|\phi_A^i\rangle\}, \{|\phi_{BC}^i\rangle\}$ are orthogonal basis for system A and system BC respectively. The density operator $\rho_{ABC} = \sum_{i,j} \sqrt{\lambda_i \lambda_j} |\phi_A^i\rangle \langle \phi_A^j| \otimes |\phi_{BC}^i\rangle \langle \phi_{BC}^j|$, the partial transpose of ρ_{ABC} with respect to system A is given by: $\rho_{ABC}^{T_A} = \sum_{i,j} \sqrt{\lambda_i \lambda_j} |\phi_A^{j*}\rangle \langle \phi_A^{i*}| \otimes |\phi_{BC}^i\rangle \langle \phi_{BC}^j|$. The negativity of $|\psi\rangle_{A|BC}$ is:

$$\begin{aligned} \mathcal{N}_{A|BC} &= \|\rho_{ABC}^{T_A}\| - 1 \\ &= \left\| \sum_{i,j} \sqrt{\lambda_i \lambda_j} |\phi_A^{j*}\rangle \langle \phi_A^{i*}| \otimes |\phi_{BC}^i\rangle \langle \phi_{BC}^j| \right\| - 1 \\ &= \left\| \sum_{i,j} \sqrt{\lambda_i \lambda_j} |\phi_A^{j*}\rangle \langle \phi_{BC}^j| \otimes |\phi_{BC}^i\rangle \langle \phi_A^{i*}| \right\| - 1 \\ &= \left\| \sum_j \sqrt{\lambda_j} |\phi_A^{j*}\rangle \langle \phi_{BC}^j| \otimes \sum_i \sqrt{\lambda_i} |\phi_{BC}^i\rangle \langle \phi_A^{i*}| \right\| - 1 \\ &= \|R \otimes R^\dagger\| - 1 \\ &= \|R\|^2 - 1 \\ &= \left(\sum_i \sqrt{\lambda_i} \right)^2 - 1 \\ &= 2\sqrt{\lambda_0 \lambda_1} \\ &= 2\sqrt{\det \rho_A} \\ &= \mathcal{C}_{A|BC}, \end{aligned} \quad (5)$$

where $R = \sum_j \sqrt{\lambda_j} |\phi_A^{j*}\rangle \langle \phi_{BC}^j|$, and we have used the property of trace norm: $\|A \otimes B\| = \|A\| \otimes \|B\|$. \square

Now we will study the monogamy property of α th power of negativity \mathcal{N}^α .

Theorem 1 . For a pure state $|\psi_{A|BC}\rangle$ in a $2 \otimes 2 \otimes 2^{N-2}$ system, the α th power of negativity satisfies the monogamy inequality:

$$\mathcal{N}_{A|BC}^\alpha \geq \mathcal{N}_{AB}^\alpha + \mathcal{N}_{AC}^\alpha, \quad (6)$$

for $\alpha \geq 2$, and satisfy the polygamy inequality:

$$\mathcal{N}_{A|BC}^\alpha < \mathcal{N}_{AB}^\alpha + \mathcal{N}_{AC}^\alpha, \quad (7)$$

for $\alpha \leq 0$.

Proof: When $\alpha \geq 2$, by using *Lemma 1*, we obtain $\mathcal{N}_{A|BC} = \mathcal{C}_{A|BC}$. Combine with the result from Re. [14]:

$$\mathcal{C}_{A|BC}^\alpha \geq \mathcal{C}_{AB}^\alpha + \mathcal{C}_{AC}^\alpha, \quad (8)$$

for $\alpha \geq 2$, We have

$$\begin{aligned} \mathcal{N}_{A|BC}^\alpha &= \mathcal{C}_{A|BC}^\alpha \\ &\geq \mathcal{C}_{AB}^\alpha + \mathcal{C}_{AC}^\alpha \\ &\geq \mathcal{N}_{AB}^\alpha + \mathcal{N}_{AC}^\alpha, \end{aligned} \quad (9)$$

the last inequality is due to for any mixed state in a $2 \otimes d$ ($2 \leq d$) quantum system, concurrence is an upper bound of negative, i.e. $\mathcal{N}_{AC} \leq \mathcal{C}_{AC}$ [16]. When $\alpha \leq 0$, without loss of generality, assuming $\mathcal{N}_{AB} \geq \mathcal{N}_{AC} > 0$, we have: $\mathcal{N}_{A|BC}^\alpha \leq (\mathcal{N}_{AB}^2 + \mathcal{N}_{AC}^2)^{\frac{\alpha}{2}} = \mathcal{N}_{AB}^\alpha (1 + \frac{\mathcal{N}_{AC}^2}{\mathcal{N}_{AB}^2})^{\frac{\alpha}{2}} < \mathcal{N}_{AB}^\alpha [1 + (\frac{\mathcal{N}_{AC}}{\mathcal{N}_{AB}})^2]^{\frac{\alpha}{2}} = \mathcal{N}_{AB}^\alpha + \mathcal{N}_{AC}^\alpha$, where we used the property for the second inequality: $(1+x)^t < 1+x^t$ ($x > 0, t \leq 0$). If $\mathcal{N}_{AB} = 0$ or $\mathcal{N}_{AC} = 0$, the inequality $\mathcal{N}_{A|BC}^\alpha < \mathcal{N}_{AB}^\alpha + \mathcal{N}_{AC}^\alpha$ obviously holds. \square

If we consider any N -qubit pure state $|\psi_{A_1 A_2 \dots A_N}\rangle$ in k -partite cases with $k = \{3, 4, \dots, N\}$. From *Theorem 1*, a set of hierarchical monogamy inequalities of \mathcal{N}^α holds:

$$\mathcal{N}_{A_1|A_2 \dots A_N}^\alpha \geq \sum_{i=2}^{k-1} \mathcal{N}_{A_1 A_i}^\alpha + \mathcal{N}_{A_1|A_k \dots A_N}^\alpha, \quad (10)$$

for $\alpha \geq 2$, and a set of hierarchical polygamy inequalities of \mathcal{N}^α holds:

$$\mathcal{N}_{A_1|A_2 \dots A_N}^\alpha \leq \sum_{i=2}^{k-1} \mathcal{N}_{A_1 A_i}^\alpha + \mathcal{N}_{A_1|A_k \dots A_N}^\alpha, \quad (11)$$

for $\alpha \leq 0$.

These set of hierarchical relations can be used to detect the multipartite entanglement in these k -partite [15]. We can also obtain the following result:

Corollary 1 . For any N -qubit pure state $|\psi_{A_1 A_2 \dots A_N}\rangle$ the general monogamous inequality hold:

$$\mathcal{N}_{A_1|A_2 \dots A_N}^\alpha \geq \mathcal{N}_{A_1 A_2}^\alpha + \dots + \mathcal{N}_{A_1 A_N}^\alpha, \quad (12)$$

for $\alpha \geq 2$, and the general polygamous inequality holds:

$$\mathcal{N}_{A_1|A_2 \dots A_N}^\alpha < \mathcal{N}_{A_1 A_2}^\alpha + \dots + \mathcal{N}_{A_1 A_N}^\alpha, \quad (13)$$

for $\alpha \leq 0$.

We can see the result of $\alpha = 2$ from Re. [10] is a special case of our monogamy inequality Eq. (12).

III. MONOGAMY OF α TH POWER CONVEX-ROOF EXTENDED NEGATIVITY

Given a bipartite state ρ_{AB} in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. CREN is defined as the convex roof extended of negativity on pure states [17]:

$$\tilde{\mathcal{N}}(\rho_{AB}) = \min \sum_i p_i \mathcal{N}(|\psi_{AB}^i\rangle), \quad (14)$$

where the minimum is taken over all possible pure state decompositions $\{p_i, \psi_{AB}^i\}$ of ρ_{AB} . Obviously, the CREN of a pure state is equal to its Negativity. CREN gives a perfect discrimination of PPT bound entangled states and separable states in any bipartite quantum systems [22, 23]. We have following result for CREN:

Theorem 2 . For a mixed state $\rho_{A|BC}$ in a $2 \otimes 2 \otimes 2^{N-2}$ system, the following monogamy inequality holds:

$$\tilde{\mathcal{N}}_{A|BC}^\alpha \geq \tilde{\mathcal{N}}_{AB}^\alpha + \tilde{\mathcal{N}}_{AC}^\alpha, \quad (15)$$

for $\alpha \geq 2$, and following polygamy inequality holds:

$$\tilde{\mathcal{N}}_{A|BC}^\alpha < \tilde{\mathcal{N}}_{AB}^\alpha + \tilde{\mathcal{N}}_{AC}^\alpha, \quad (16)$$

for $\alpha \leq 0$.

Proof: We only prove the first monogamy inequality, the proof of second inequality is similar to the proof of *Theorem 1*. Assuming a mixed state $\rho_{A|BC}$ in a $2 \otimes 2 \otimes 2^{N-2}$ system, by using the *Lemma 1*, the definition of CREN and concurrence, we have:

$$\begin{aligned} \tilde{\mathcal{N}}_{A|BC} &= \min \sum_i p_i \mathcal{N}(|\psi_{A|BC}^i\rangle) \\ &= \min \sum_i p_i \mathcal{C}(|\psi_{A|BC}^i\rangle) \\ &= \mathcal{C}_{A|BC}. \end{aligned} \quad (17)$$

Thus we have:

$$\begin{aligned} \tilde{\mathcal{N}}_{A|BC}^\alpha &= \mathcal{C}_{A|BC}^\alpha \\ &\geq \mathcal{C}_{AB}^\alpha + \mathcal{C}_{AC}^\alpha \\ &\geq \tilde{\mathcal{N}}_{AB}^\alpha + \tilde{\mathcal{N}}_{AC}^\alpha, \end{aligned} \quad (18)$$

for $\alpha \geq 2$, where the second inequality is due to for any mixed state in a $2 \otimes d$ ($2 \leq d$) quantum system, concurrence is an upper bound of negative. \square

From *Theorem 2*, a set of hierarchical monogamy inequalities of $\tilde{\mathcal{N}}^\alpha$ holds for any N -qubit mixed state $\rho_{A_1 A_2 \dots A_N}$ in k -partite cases with $k = \{3, 4, \dots, N\}$:

$$\tilde{\mathcal{N}}_{A_1|A_2 \dots A_N}^\alpha \geq \sum_{i=2}^{k-1} \tilde{\mathcal{N}}_{A_1 A_i}^\alpha + \tilde{\mathcal{N}}_{A_1|A_k \dots A_N}^\alpha, \quad (19)$$

for $\alpha \geq 2$, and a set of hierarchical polygamy inequalities of $\tilde{\mathcal{N}}^\alpha$ holds:

$$\tilde{\mathcal{N}}_{A_1|A_2 \dots A_N}^\alpha \leq \sum_{i=2}^{k-1} \tilde{\mathcal{N}}_{A_1 A_i}^\alpha + \tilde{\mathcal{N}}_{A_1|A_k \dots A_N}^\alpha, \quad (20)$$

for $\alpha \leq 0$.

We also have the following corollary:

Corollary 2 . For a mixed state $\rho_{A_1 A_2 \dots A_N}$ in a N -qubit system, the α th power of CREN satisfies:

$$\tilde{\mathcal{N}}_{A_1|A_2 \dots A_N}^\alpha \geq \tilde{\mathcal{N}}_{A_1 A_2}^\alpha + \dots + \tilde{\mathcal{N}}_{A_1 A_N}^\alpha, \quad (21)$$

for $\alpha \geq 2$ and

$$\tilde{\mathcal{N}}_{A_1|A_2 \dots A_N}^\alpha < \tilde{\mathcal{N}}_{A_1 A_2}^\alpha + \dots + \tilde{\mathcal{N}}_{A_1 A_N}^\alpha, \quad (22)$$

for $\alpha \leq 0$.

We can see the result of $\alpha = 2$ from Re. [11] is a special case of our monogamy inequality Eq. (21).

IV. MONOGAMY OF α TH POWER OF ENTANGLEMENT OF FORMATION

Given a bipartite state ρ_{AB} in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the entanglement of formation (EoF) is defined as [18, 19]:

$$E(\rho_{AB}) = \min \sum_i p_i E(|\psi_{AB}^i\rangle), \quad (23)$$

where $E(|\psi_{AB}^i\rangle) = -\text{Tr} \rho_A^i \log_2 \rho_A^i = -\text{Tr} \rho_B^i \log_2 \rho_B^i$ is the von Neumann entropy, the minimum is taken over all possible pure state decompositions $\{p_i, \psi_{AB}^i\}$ of ρ_{AB} . In Re. [20], Wootters derived an analytical formula for a two-qubit mixed state ρ_{AB} :

$$E(\rho_{AB}) = h\left(\frac{1 + \sqrt{1 - \mathcal{C}_{AB}^2}}{2}\right), \quad (24)$$

where $h(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ is the binary entropy and \mathcal{C}_{AB} is the concurrence of ρ_{AB} which is given by Eq. (3) and Eq. (4). Bai *et al* have proven a set of hierarchical monogamy inequalities holds for the squared EoF in a $2 \otimes 2 \otimes 2^{N-2}$ system [15].

$$E_{A_1|A_2 \dots A_N}^2 \geq \sum_{i=2}^{k-1} E_{A_1 A_i}^2 + E_{A_1|A_k \dots A_N}^2. \quad (25)$$

We will show that the hierarchical monogamy inequality holds for the α th power of EoF, where $\alpha \geq \sqrt{2}$. Our result can be seen an improvement of Bai *et al*'s work.

Theorem 3 . For a mixed state $\rho_{A|BC}$ in a $2 \otimes 2 \otimes 2^{N-2}$ system, the following monogamy inequality for the α th power of EoF holds:

$$E_{A|BC}^\alpha \geq E_{AB}^\alpha + E_{AC}^\alpha, \quad (26)$$

for $\alpha \geq \sqrt{2}$, and the following polygamy inequality holds:

$$E_{A|BC}^\alpha \leq E_{AB}^\alpha + E_{AC}^\alpha, \quad (27)$$

for $\alpha \leq 0$.

Proof: Let's consider a tripartite pure state $|\phi_{ABC}\rangle$ in a $2 \otimes 2 \otimes 2^{N-2}$ system. Based on the Schmidt decomposition, the 2^{N-2} -dimensional qubit **C** can be viewed as

an *effect* four-dimensional qubit [3]. Therefore, we can consider the monogamy relationship in a $2 \otimes 2 \otimes 4$ system:

$$\begin{aligned} E_{A|BC}^\alpha &= E^\alpha(\mathcal{C}_{A|BC}^2) \\ &\geq E^\alpha(\mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2) \\ &\geq E^\alpha(\mathcal{C}_{AB}^2) + E^\alpha(\mathcal{C}_{AC}^2) \\ &= E^\alpha(\rho_{AB}) + E^\alpha(\rho_{AC}), \end{aligned} \quad (28)$$

where the first inequality is due to $E(\mathcal{C}^2)$ is a monotonic increasing function and $\mathcal{C}_{A|BC}^2 \geq \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2$ holds, the second inequality is due to the fact [14]: $E^\alpha(\mathcal{C}_1^2 + \mathcal{C}_2^2) \geq E^\alpha(\mathcal{C}_1^2) + E^\alpha(\mathcal{C}_2^2)$ for all $\alpha \geq \sqrt{2}$, the last equality is due to a mixed state ρ_{AC} in a $2 \otimes d$ system, $E(\rho_{AC}) = E(\mathcal{C}^2(\rho_{AC}))$ [15]. Thus, we complete our discussion on pure state.

Consider a mixed state ρ_{ABC} in a $2 \otimes 2 \otimes 2^{N-2}$ system. We use an optimal convex decomposition $\{p_i, |\phi_{ABC}^i\rangle\}$:

$$E(\rho_{A|BC}) = \sum_i p_i E(|\phi_{ABC}^i\rangle), \quad (29)$$

we can derive

$$\begin{aligned} E(\rho_{A|BC}) &= \sum_i p_i E(|\phi_{ABC}^i\rangle) \\ &= \sum_i p_i E[\mathcal{C}^2(|\phi_{ABC}^i\rangle)] \\ &\geq E[\sum_i p_i \mathcal{C}^2(|\phi_{ABC}^i\rangle)] \\ &\geq E[\mathcal{C}^2(\rho_{A|BC})] \\ &\geq \sqrt[\alpha]{E^\alpha(\mathcal{C}_{AB}^2) + E^\alpha(\mathcal{C}_{AC}^2)} \\ &= E^\alpha(\rho_{AB}) + E^\alpha(\rho_{AC}), \end{aligned} \quad (30)$$

where the first equality is the definition of mixed state, we have used that $E(\mathcal{C}^2)$ is a convex function in the first inequality, the second inequality can be derived by Cauchy-Schwarz inequality: $(\sum_i x_i^2)^{\frac{1}{2}} (\sum_i y_i^2)^{\frac{1}{2}} \geq \sum_i x_i y_i$, with $x_i = \sqrt{p_i}$, $y_i = \sqrt{p_i} \mathcal{C}^2(|\phi_{ABC}^i\rangle)$. Thus proving the monogamy inequality. On the other hand, it is easy to check the polygamy inequality for $\alpha \leq 0$. \square

Based on the discussion above, we show that for a mixed state $\rho_{A_1 A_2 \dots A_N}$ in a $2 \otimes 2 \otimes 2^{N-2}$ system, a set of hierarchical monogamy inequalities holds for the α th power of EoF in k -partite case with $k = \{3, 4, \dots, N\}$:

$$E_{A_1|A_2 \dots A_N}^\alpha \geq \sum_{i=2}^{k-1} E_{A_1 A_i}^\alpha + E_{A_1|A_k \dots A_N}^\alpha, \quad (31)$$

for $\alpha \geq \sqrt{2}$, which can be an improvement of Bai *et al.*'s work. And a set of hierarchical polygamy inequalities holds:

$$E_{A_1|A_2 \dots A_N}^\alpha \leq \sum_{i=2}^{k-1} E_{A_1 A_i}^\alpha + E_{A_1|A_k \dots A_N}^\alpha, \quad (32)$$

for $\alpha \leq 0$. When $k = N$, the general monogamy inequality hold:

$$E_{A_1|A_2 \dots A_N}^\alpha \geq E_{A_1 A_2}^\alpha + \dots + E_{A_1 A_N}^\alpha, \quad (33)$$

for $\alpha \geq \sqrt{2}$, the specific case have been revealed in Re. [14]. We also have the general polygamy inequality:

$$E_{A_1|A_2 \dots A_N}^\alpha \leq E_{A_1 A_2}^\alpha + \dots + E_{A_1 A_N}^\alpha, \quad (34)$$

for $\alpha \leq 0$.

V. MONOGAMY OF α TH POWER CONCURRENCE VS MONOGAMY OF α TH POWER NEGATIVITY

Based on the monogamy inequality of concurrence [2, 3], Re. [14] considered the general monogamy inequalities of α th power concurrence in an N -qubit mixed state $\rho_{A_1 A_2 \dots A_N}$, and claimed the following inequalities holds:

$$\mathcal{C}_{A_1|A_2 \dots A_N}^\alpha \geq \mathcal{C}_{A_1 A_2}^\alpha + \dots + \mathcal{C}_{A_1 A_N}^\alpha \quad (35)$$

for $\alpha \geq 2$. While the polygamy inequalities holds:

$$\mathcal{C}_{A_1|A_2 \dots A_N}^\alpha < \mathcal{C}_{A_1 A_2}^\alpha + \dots + \mathcal{C}_{A_1 A_N}^\alpha. \quad (36)$$

for all $\alpha \leq 0$. It's not clear for $0 < \alpha < 2$. In this section, we will discuss the monogamy property of α th power of concurrence and α th power of negativity, for $0 < \alpha < 2$. For convenience, we define the "residual tangle" of α th power of concurrence as:

$$\tau^{\mathcal{C}}(|\psi_{A_1 A_2 \dots A_N}\rangle) = \mathcal{C}_{A_1 A_2 \dots A_N}^\alpha - \mathcal{C}_{A_1 A_2}^\alpha - \dots - \mathcal{C}_{A_1 A_N}^\alpha, \quad (37)$$

and define the "residual tangle" of α th power of concurrence as:

$$\tau^{\mathcal{N}}(|\psi_{A_1 A_2 \dots A_N}\rangle) = \mathcal{N}_{A_1 A_2 \dots A_N}^\alpha - \mathcal{N}_{A_1 A_2}^\alpha - \dots - \mathcal{N}_{A_1 A_N}^\alpha. \quad (38)$$

Interestingly, We find that the N -qubit GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}), \quad (39)$$

and N -qubit W state

$$|W\rangle = \frac{1}{\sqrt{N}}(|00 \dots 01\rangle + |00 \dots 10\rangle + \dots + |10 \dots 00\rangle), \quad (40)$$

can be used to distinguish the monogamous property of $\tau^{\mathcal{C}}(|\psi_{A_1 A_2 \dots A_N}\rangle)$ for $0 < \alpha < 2$. In other words, N -qubit GHZ state is monogamous for the α th power concurrence and N -qubit W state is polygamous for the α th power concurrence, where $0 < \alpha < 2$. For N -qubit GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}), \quad (41)$$

the concurrence $\mathcal{C}_{A_1 A_2 \dots A_N} = 1, \mathcal{C}_{A_1 A_k} = 0, k = \{2, 3, \dots, N\}$. Thus, the "residual tangle" $\tau^{\mathcal{C}}(|GHZ\rangle) =$

$1 > 0$, N -qubit GHZ state is monogamous for the α th power concurrence. For N -qubit W state

$$|W\rangle = \frac{1}{\sqrt{N}}(|00\cdots 01\rangle + |00\cdots 10\rangle + \cdots + |10\cdots 00\rangle), \quad (42)$$

the concurrence $\mathcal{C}_{A_1 A_2 \dots A_N} = \frac{2}{N}\sqrt{N-1}$, $\mathcal{C}_{A_1 A_k} = \frac{2}{N}$, $k = \{2, 3, \dots, N\}$. Thus, the "residual tangle" $\tau^{\mathcal{C}}(|W\rangle) = (\frac{2}{N})^\alpha [(N-1)^{\frac{\alpha}{2}} - (N-1)] < 0$ for all $0 < \alpha < 2$. N -qubit W state is polygamous for the α th power concurrence.

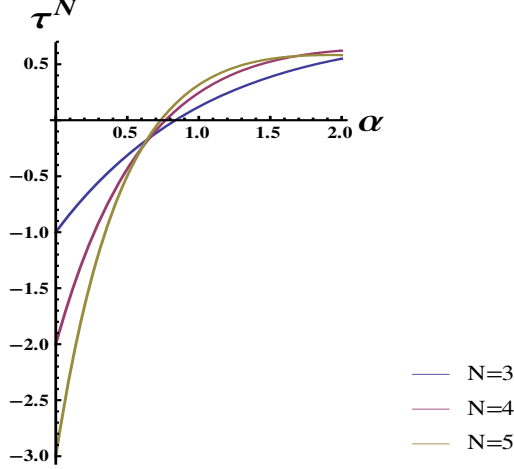


FIG. 1: (Color online) $\tau^N(|W\rangle)$ as the function of α for $0 < \alpha < 2$, the blue line, red line and yellow line denote $N = 3$, $N = 4$ and $N = 5$ respectively.

For the "residual tangle" $\tau^{\mathcal{N}}(|\psi_{A_1 A_2 \dots A_N}\rangle)$. The negativity of N -qubit GHZ state $\mathcal{N}_{A_1 A_2 \dots A_N} = 1$, $\mathcal{N}_{A_1 A_k} = 0$, $k = \{2, 3, \dots, N\}$. Thus, $\tau^{\mathcal{N}}(|GHZ\rangle) = 1 > 0$, it is coincide with $\tau^{\mathcal{C}}(|GHZ\rangle)$. The situation is different

when we consider $\tau^{\mathcal{C}}(|\psi_{A_1 A_2 \dots A_N}\rangle)$ for N -qubit W state. One obtain that $\mathcal{N}_{A_1 A_2 \dots A_N} = \frac{2}{N}\sqrt{N-1}$ and $\mathcal{N}_{A_1 A_k} = \frac{1}{N}\sqrt{2(N-2)^2 + 4 - 2(N-2)\sqrt{(N-2)^2 + 4}}$, $k = \{2, 3, \dots, N\}$. It is easy to check that $\tau^{\mathcal{N}}(|W\rangle) = \frac{1}{N^\alpha} [2^\alpha (N-1)^{\frac{\alpha}{2}} - (N-1)[2(N-2)^2 + 4 - 2(N-2)\sqrt{(N-2)^2 + 4}]^\alpha$. $\tau^{\mathcal{N}}(|W\rangle)$ can be positive and negative, as showed in Fig:1, we have plotted $\tau^{\mathcal{N}}(|W\rangle)$ as the function of α for $0 < \alpha < 2$, and consider $N = 3$, $N = 4$ and $N = 5$ respectively. We find $\tau^{\mathcal{N}}(|W\rangle)$ is not always negative, which is different than the case of $\tau^{\mathcal{C}}(|W\rangle)$.

VI. CONCLUSION

In this paper, We studied the monogamy property of α th power of entanglement measure in bipartite states. In particular, we investigated the monogamy properties of negativity and CREN in detail. We showed that the α th power of negativity, CREN are monogamous for $\alpha \geq 2$ and polygamous for $\alpha \leq 0$. We improved the hierarchical monogamy inequality for the α th power of EoF, and show that the α th power of EoF is hierarchical monogamous for $\alpha \geq \sqrt{2}$. Finally, we discussed the monogamy property of α th power of concurrence. We found the N -qubit GHZ state and N -qubit W state can be used to distinguish the α th power the concurrence for $0 < \alpha < 2$ in qubit system. We compared concurrence with negativity in terms of monogamy property and showed the difference between them.

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